Tutorial 8

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10th week

1. Calculate area with green's theorem.

   Green's Theorem Area Formula

$$\text{Area of } R = \frac{1}{2}\oint_C x\mathrm{d}y - y\mathrm{d}x$$

   Use this theorem to calculate the area enclosed by $\vec{r}(t) = (\cos^3 t)\vec{i} + (\sin^3 t)\vec{j}$

The reason of this formula

$$\text{Area of } R = \iint_R dy\,dx = \iint_R \left(\tfrac{1}{2}+\tfrac{1}{2}\right) dy\,dx$$

$$= \oint_C \tfrac{1}{2}x\,dy - \tfrac{1}{2}y\,dx$$

The curve $C: \begin{cases} x = \cos^3 t \\ y = \sin^3 t \end{cases}$ $0 \le t \le 2\pi$

$dy = 3\sin^2 t \cos t\,dt$ $dx = -3\cos^2 t \sin t\,dt$

$$\text{Area of } R = \frac{1}{2}\int_0^{2\pi} 3\cos^3 t \sin^2 t \cos t + 3\sin^3 t \cos^2 t \sin t\,dt$$

$$= \frac{3}{2}\int_0^{2\pi} \cos^4 t \sin^2 t + \sin^4 t \cos^2 t\,dt$$

$$= \frac{3}{2}\int_0^{2\pi} \cos^2 t \sin^2 t\,dt$$

$$= \frac{3}{2}\int_0^{2\pi} \cos^2 t(1-\cos^2 t)\,dt$$

$$= \frac{3}{2}\left[\int_0^{2\pi} \frac{1+\cos 2t}{2}\,dt - \int_0^{2\pi} \cos^4 t\,dt\right]$$

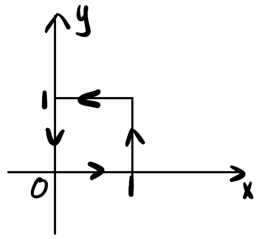
$$= \frac{3}{2}\left[\int_0^{2\pi} \frac{1}{2}+\frac{1}{2}\cos 2t\,dt - \int_0^{2\pi} \frac{3}{8}+\frac{1}{2}\cos 2t+\frac{1}{8}\cos 4t\,dt\right.$$

$$= \frac{3}{8}\pi$$

2. Use Green's Theorem to find the counter-clockwise circulation and outward flux for the field $\vec{F}$ and curve C

$$\vec{F} = (x^2 + 4y)\vec{i} + (x + y^2)\vec{j}$$

C: The square bounded by $x = 0, x = 1, y = 0, y = 1$.



Circulation curl

$$\oint_C (x^2+4y)dx + (x+y^2)dy$$

$$= \iint_R \frac{\partial (x+y^2)}{\partial x} - \frac{\partial (x^2+4y)}{\partial y} \, dy\,dx$$

$$= \int_0^1 \int_0^1 -3 \quad dy\,dx$$

$$= -3$$

Outward flux

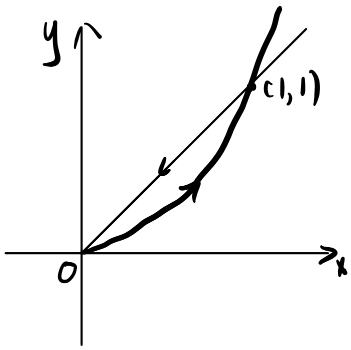
$$\oint_C F \cdot \vec{n}\, d\vec{r} = \iint_R \left(\frac{\partial (x^2+4y)}{\partial x} + \frac{\partial (x+y^2)}{\partial y} \right) dy\,dx$$

$$= \iint_R (2x + 2y) dy\,dx$$

$$= 2\int_0^1 \int_0^1 x+y \, dy\,dx$$

$$= 2$$

3. Find the counter-clockwise circulation and outward flux of the field $\vec{F} = xy\vec{i} + y^2\vec{j}$ around and over the boundary of the region enclosed by the curves $y = x^2$ and $y = x$ in the first quadrant.



Circulation curl

$$\oint_C xy\,dx + y^2\,dy$$

$$= \iint_R \frac{\partial y^2}{\partial x} - \frac{\partial xy}{\partial y}\,dy\,dx$$

$$= \int_0^1 \int_{x^2}^{x} -x \quad dy\,dx$$

$$= -\frac{1}{12}$$

Outward flux

$$\oint_C \vec{F}\cdot\vec{n}\,dx = \iint_R \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\,dy\,dx$$

$$= \iint_R y + 2y\,dy\,dx$$

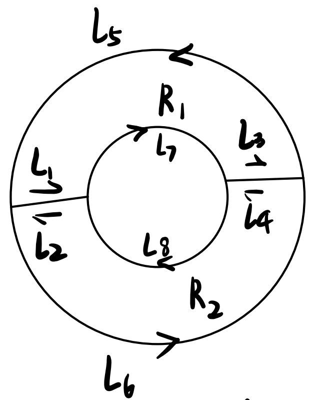
$$= 3\int_0^1 \int_{x^2}^{x} y\,dy\,dx$$

$$= \frac{1}{5}$$

outer circle counter-clockwise

inner circle clockwise.

4. Evaluate $\oint_C y^3dx - x^3dy$ where C are the two circles of radius 2 and radius 1 centered at the origin with positive orientation.



L_1: line segment from $(-2,0)$ to $(-1,0)$

⋮

$C_1 = L_1 \cup L_7 \cup L_3 \cup L_5$

$C_2 = L_2 \cup L_6 \cup L_4 \cup L_8$

$$\oint_C y^3dx - x^3dy$$

$$= \oint_{C_1} y^3dx - x^3dy + \oint_{C_2} y^3dx - x^3dy$$

$$= \iint_{R_1} -3(x^2+y^2)\,dydx + \iint_{R_2} -3(x^2+y^2)\,dydx$$

$$= -3\iint_R (x^2+y^2)\,dydx$$

$$= -3\int_0^{2\pi}\int_1^2 r^2 \cdot r\,dr\,d\theta$$

$$= -\frac{45\pi}{2}$$